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ABSTRACT

A mathematical model for computer-aided instruction has been developed. The assumption is made that the course is divided into a hierarchy of levels of difficulty and that if a student is able to perform successfully at a given level of difficulty, he can also perform successfully at all levels of lesser difficulty. Furthermore, if a student performs successfully at one level, the probability of successful performance at the next higher level is increased. Given the initial vector of probabilities for successful performance at each level, and the expected times it takes to attempt a successful performance at each level, the model computes an instructional sequence which minimizes the expected time required for the student to complete the course by performing successfully at the highest level of difficulty. Dynamic programing is used to find this sequence. (Author)

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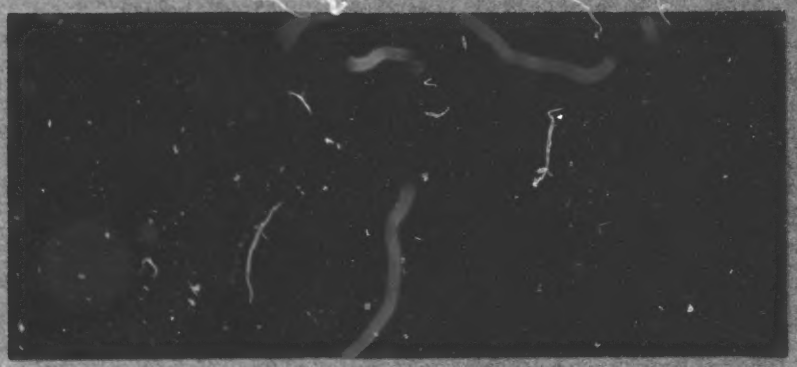

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University of Southern California

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Technical Report No. 72

A MARKOV DECISION MODEL FOR COMPUTER-AIDED INSTRUCTION

December 1973

Richard D. Wollmer

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ABSTRACT

A mathematical model for computer aided instruction is developed. It is assumed that the course is divided into a hierarchy of levels of difficulty. These levels are such that if a student is able to perform successfully at a given level of difficulty, he can also perform successfully at all levels of lesser difficulty. Furthermore, if a student performs successfully at one level, it increases his probability of being able to perform successfully at the next higher level of difficulty. Given the initial vector of probabilities for successful performance at each level, the vector describing how these probabilities change with successful performances at each level, and the expected times it takes to attempt a successful performance at each level, this model computes an instructional sequence that minimizes the expected time required for the student to complete the course by performing successfully at the highest level of difficulty. Dynamic programming is used to find this sequence.

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A MARKOV DECISION MODEL FOR COMPUTER-AIDED INSTRUCTION

I. INTRODUCTION

Several researchers have been interested in the application of optimization techniques to models of learning and instruction. Karush and Dear (1966) developed an optimal strategy for teaching students to learn a list of independent items. The basic assumption was that an item is either in a learned or unlearned state. If it is given in the unlearned state it goes to the learned state with probability c, while once it reaches the learned state it stays there. Atkinson and Paulson (1972) described experiments in which extensions of this model were applied to computer-assisted spelling instruction with elementary school childrer. Chant and Atkinson (1973) developed an optimization technique for allocating instructional effort to two interrelated blocks or strands of learning material. Their key assumption was that the learning rate for each of the two strands depends solely on the difference between the achievement levels on the two strands.

The model of this report concerns a system where a student is to be taught to perform a certain skill at a given level of competence. He achieves this by working problems thro ugh or taking tests at various levels of difficulty. It is assumed that

if a student is able to perform successfully at one level of difficulty he is able to perform at the next lower or preceding level of difficulty and consequently at all lower levels of difficulty. This assumption is particularly applicable in the following two situations.

The first situation is one where the material covered at one level includes all that covered at preceding levels, plus some additional material. An example of this is a program developed at Behavioral Technology Laboratories to teach students Kirchoff's laws. This course is comprised of eleven levels with the lowest level defining the units for voltage, current and resistance up to the highest level which deals with the application of Ohm's law and Kirchoff's voltage and current laws in complex networks.

The second situation is one where the material and problems covered at a particular level are virtually the same as at the immediately preceding level except more clues and hints are given at the preceding level. A good example of this would be a version of the Kirchoff's laws program considered earlier at Behavioral Technology Laboratories in which problems would be given at the following levels:

1. Problems are given in steps with cues and knowledge of results at each step.
2. Problems are given in steps with no cues or knowledge of results at each step
3. The student solves problems in steps but he chooses the steps.

4. The student is simply given problems and asked to solve them.

Note, however, the assumption given for this model would not be applicable for the situation where a given level did not use certain material introduced at preceding levels.

It is also assumed that if a student performs successfully at one level, it will increase his probability of being able to perform successfully at the next higher level. The student completes the course when he performs successfully at the highest level. The aim of the model presented in this paper is to choose the levels at which problems should be assigned in the course sequence so the expected time required by the student to complete the course is minimized.

II. THE MODEL

Mathematical Formulation

The problem of instructing the student so that he completes the course in minimum time is formulated as a Markov decision process. The set of actions are $1, \ldots, N$ where action i is that of giving the student a problem at level i. The levels are numbered in decreasing order of difficulty. Thus level 1 is the hardest and level N the easiest. The state ϕ is that in which the student has performed successfully at level 1. The states in which the student has not performed successfully at level 1 are characterized by the vectors $p = (p_1, \ldots, p_n)$ where p_i equals the probability that the student will correctly do a problem at level i. It is assumed

that if a student can do a problem at level i, he can also do it at level j for all $j > i$. Thus, p_i is non-decreasing in i.

For each action i, let

$q_i = P$ [student can perform at level $i - 1$ / student completes problem at level i correctly and could not perform at level $i - 1$ before].

Thus, if the state is p and we perform action 1, we go to state ϕ with probability p_1 and remain in state p with probability $(1 - p_1)$. If we take action $i > 1$ we go to state $\bar{p}$ with probability p_i where $\bar{p}_{i-1} = p_{i-1} + q_i(1 - p_{i-1})$, $\bar{p}_k = p_k$, $k \neq i - 1$, and remain in state p with probability $(1 - p_i)$.

Equivalently, the components of $\bar{p}$ above may be represented by the following:

$$\bar{p}_{i-1} = (1 - q_i)\, p_{i-1} + q_i$$

$$\bar{p}_k = p_k, \quad k \neq i-1 \quad . \tag{1}$$

Once the system reaches state ϕ, it remains there. Associated with each action, i, is a cost c_i which may be equal to the expected time it takes to attempt a problem at level i. It is desired to choose an action policy that reaches state ϕ at minimum cost.

Some Solution Properties

A policy specifies an action for each state of the system other than state ϕ. Let $V(\pi, p)$ be the total expected cost under policy π when the system is in state p. If i is the action specified for state p by π, then it follows that:

$$V(\pi, p) = c_i + p_i V(\pi, \bar{p}) + (1 - p_i)V(\pi, p) \tag{2}$$

where $\bar{p}_{i-1} = (1-q_i)\, p_{i-1} + q_i$ and $\bar{p}_k = p_k$ for $k \neq i-1$.

It is of course desired to find π so that $V(\pi, p) \leq V(\bar{\pi}, p)$ for all p and all $\bar{\pi}$.

Note that if action i is taken and the student does not complete his task at level i successfully, the state does not change and action i will be repeated. Thus any action taken will be repeated until the student completes a problem correctly at which time the state changes and a new action may be taken. In addition, the state resulting from the first time the student performs correctly at level i is independent of the number of attempts it takes the student to perform successfully at that level. Thus π and p determine a sequence of correct responses at each level though not the number of trials necessary to obtain these responses. Of course, the sequence must end with one correct response at level 1. The expected number of trials necessary for the student to complete a problem successfully at level i is $1/p_i$ and the expected cost of this is c_i/p_i.

Consider the policy that requires performance at level 1 only for p. The cost of this policy would be c_1/p_1 which would be less than that of any policy requiring more than $c_1/c_i p_1$ successful performances at level i. Hence, the set of performance sequences for p that are superior to testing at level 1 only is finite and there is an optimal sequence for each p. This establishes the existence of a policy π satisfying $V(\pi, p) \leq V(\bar{\pi}, p)$ all p and all $\bar{\pi}$.

We are now ready to prove the following theorems.

Theorem 1: If $p - r \geq 0$, then $V(\pi, p) \leq V(\pi, r)$, if π is an optimal policy.

Proof: If π consists of one correct response (at level 1) for both p and r, then $V(\pi, p) = c_1/p_1$ while $V(\pi, r) = c_1/r_1$ and the theorem holds. Suppose the theorem holds for all p and r such that π specifies n or fewer total correct responses for r. Consider p and r such that π requires $n+1$ or fewer correct responses for r and let i be the level at which the first correct response for state r must take place. Then $V(\pi, p) \leq c_i/p_i + V(\pi, \bar{p})$ and $V(\pi, r) = c_i/r_i + V(\pi, \bar{r})$ where from (1) $\bar{p}_{i-1} = (1 - q_i)\, p_{i-1} + q_i$, $\bar{r}_{i-1} = (1 - q_i)\, r_{i-1} + q_i$, $\bar{p}_k = p_k$ and $\bar{r}_k = r_k$ for $k \neq i-1$. Thus $\bar{p} \geq \bar{r}$, $\bar{r}$ requires n or fewer correct responses and $V(\pi, p) \leq V(\pi, r)$. The theorem follows from induction.

Theorem 2: There is an optimal policy π such that if $a_1, a_2, \ldots, a_n$ is the sequence of the levels of correct responses for state p, then $a_k \geq a_{k+1}$ all k.

Proof: Let π be an optimal policy and $a_1, a_2, \ldots, a_n$ be the sequence specified for p. Suppose there is a k such that $a_k < a_{k+1}$. Let $\bar{p}$ be the state and $\bar{c}$ the expected cost resulting from the correct responses to the sequence $a_1, a_2, \ldots, a_{k-1}$ for the initial state p. Consider the policy $\bar{\pi}$ which differs from π only in that a_k and a_{k+1} are interchanged in the sequence for p and let $i = a_{k+1}$, $j = a_k$.

If $j < i-1$, then $V(\pi, p) = V(\bar{\pi}, p) = c + c_i/\bar{p}_i + c_j/\bar{p}_j + V(\pi, r)$ where from (1), $r_{i-1} = (1-q_i)\,\bar{p}_{i-1} + q_i$, $r_{j-1} = (1-q_j)\,\bar{p}_{j-1} + q_j$, $r_\ell = \bar{p}_\ell$ all other ℓ and consequently $V(\pi, p) = V(\bar{\pi}, p)$. Also from (1), if $j = i-1$, then $V(\pi, p) = \bar{c} + c_i/\bar{p}_i + c_{i-1}/\bar{p}_{i-1} + V(\pi, r)$, where r is defined above while $V(\bar{\pi}, p) = \bar{c} + c_i/p_i + c_{i-1}/[(1-q_i)\bar{p}_{i-1} + q_i] + V(\pi, r)$. Thus $V(\bar{\pi}, p) \leq V(\pi, p)$ and $\bar{\pi}$ is also optimal. Continuing in this manner, an optimal sequence for p is eventually obtained in which the members of the sequence are in non-increasing order. The theorem follows since p is general.

Thus the search for an optimal policy may be confined to those which yield a sequence of correct responses at levels that are non-increasing in the sequence.

As noted before, if one elicits one correct response at level i, p_{i-1} is transformed to $(1-q_i)\,p_{i-1} + q_i$. Applying this transformation recursively it follows that if one elicits k correct responses at level i, p_{i-1} is transformed to $(1-q_i)^k\,p_{i-1} + (1-q_i)^{k-1}\,q_i + \cdots + q_i$ which sums to $1 - (1-q_i)^k\,(1-p_{i-1})$. Thus, if π is a policy of the above type and specifies $k(i)$ correct responses at level i for p, then

$$V(\pi, p) = \sum_{i=1}^{N} k(i)\, c_i/\bar{p}_i \quad \text{where}$$

$$\bar{p}_N = p_N \tag{3}$$

$$\bar{p}_i = 1-(1-p_i)\,(1-q_{i+1})^{k(i+1)} \quad \text{for} \quad i < N$$

Let $V_n(p)$ be the minimum cost for state p if we restrict instruction to levels $1, \ldots, n$. That is no instruction takes place at levels $n+1, n+2, \ldots, N$. Of course, only the first n components of p are relevant in determining $V_n(p)$ and throughout the remainder of this paper it will be assumed that p is restricted to $p_1, \ldots, p_n$ in $V_n(p)$.

In other parts of this paper, the symbol p will be used to represent restrictions of p to certain components where the restriction is obvious. In particular, in the expression $V_n(p, p_n)$, p represents the restriction of p to $p_1, p_2, \ldots, p_{n-1}$.

From Theorem 2 and (3) if follows that

$$V_n(p) = \min_k \left[V_n^k(p) \right] \quad \text{where}$$

$$V_n^k(p) = k\, c_n/p_n + V_{n-1}\left(p,\ 1-(1-q_n)^k(1-p_{n-1})\right) \tag{4}$$

$$V_1(p) = c_1/p_1 \ .$$

Note that in the right hand side of the second line of (4) p represents the restriction of p to $p_1, p_2, \ldots, p_{n-2}$.

Algorithm for Two Levels

For the two levels problem it follows from (4) that

$$V_2(p) = \min_k \left[V_2^k(p) \right] \quad \text{where}$$

$$V_2^k(p) = k\, c_2/p_2 + c_1/\left(1 - (1-q_2)^k (1-p_1)\right) \tag{5}$$

Consequently,

$$V_2^k(p) - V_2^{k-1}(p) = c_2/p_2 + c_1/(1-(1-q_2)^k(1-p_1)) - c_1/(1-(1-q_2)^{k-1}(1-p_1)), \quad (6)$$

$V_2^k(p) \leq V_2^{k-1}(p)$ if and only if the expression in (6) is less than or equal to zero. This happens if and only if

$p_2 \geq f(k, p_1)$ where

$$f(k, p_1) = c_2 \left[(1-q_2)^k (1-p_1) - (2-q_2) + \frac{1}{(1-q_2)^{k-1}(1-p_1)} \right] \Big/ c_1 q_2 \quad (7)$$

for $k = 1, 2 \ldots$

This, requiring a student to perform successfully k times at level 2 is preferred to requiring him to perform successfully k-1 times at level 2 if $p_2 > f(k, p_1)$, while requiring him to perform successfully k-1 time at level 2 is preferred if $p_2 < f(k, p_1)$, and these two strategies yield equal costs if $p_2 = f(k, p_1)$.

Theorem 3: In (7), $f(k, p_1)$ is nondecreasing in k and is non-negative in all k.

Proof: Substituting in (7) and rearranging terms one obtains $f(1, p_1) = c_2 \left[p_1 q_2 + (1 - (1-p_1^2)) / (1-p_1) \right] / c_1 q_2$ which is clearly non-negative. Also, $f(k, p_1) - f(k-1, p_1) = c_2 \left[1/(1-q_2)^{k-1}(1-p_1) - (1-q_2)^{k-1}(1-p_1) \right] / c_1$ which is non-negative since the positive term of the second factor in the numerator exceeds 1 while the negative one is less than 1.

Thus, $f(k, p_1)$ is increasing in k. Q.E.D.

Theorem 4: Define $f(0, p_1) = 0$ and $f(k, p_1)$ as in (7) for $k = 1, 2, \ldots, m$, where m is such that $f(m, p_1) \geq 1$ and $f(m-1, p_1) < 1$. Then the value of k that minimizes (5) is that which satisfies $f(k, p_1) \leq p_2 < f(k+1, p_1)$.

Proof: If $i < k$, it follows immediately from Theorem 3 that $V_2^k(p) < V_2^{k-1}(p) < \ldots < V_2^1(p)$ and similarly if $i > k$, $V_2^k(p) < V_2^{k+1}(p) < \ldots < V_2^i(p)$.

Thus from Theorem 4, for fixed p_1, the number of successful performances required at level 2 to minimize cost is an increasing step function of p_2, starting at 0 for $p_2 = 0$ and advancing in increments of one. The minimizing cost may be found from (5) once k is known.

Additional Solution Properties

In calculating $V_n(p)$, it is much more difficult to get a closed form such as that for $V_1(p)$ and $V_2(p)$. However, as this section will show, for fixed $p_{n-1}, p_{n-2}, \ldots, p_1$, the value of k that minimizes (4) is a non-decreasing step function in p_n with a value of 0 at $p_n = 0$. However, the increments of the step function are not necessarily one. The next lemma and two theorems show this. First, define

$$f_n(j,k,p) = (k-j)\, c_n \Big/ \left[V_{n-1}(p, 1-(1-q_n)^j (1-p_{n-1})) - V_{n-1}(p, 1-(1-q_n)^k (1-p_n)) \right] \quad (8)$$

for $j < k$, $p = (p_2, \ldots, p_n)$.

Lemma 5: In (8), $f_n(j,k,p) > 0$ when defined and $V_n^j(p) > V_n^k(p)$ for $p_n < f_n(j,k,p)$; $V_n^j(p) < V_n^k(p)$ for $p_n > f_n(j,k,p)$; and equality holds for $p_n = f_n(j,k,p)$.

Proof: The denominator of $f_n(j,k,p)$ is non-negative by Theorem 1. The theorem then follows from the definition of $V_n^k(p)$ in (4).

Theorem 6: The number of successful performances required at level n in order to realize $V_n(p)$ is non-decreasing in p_n.

Proof: For $i = 1$ the theorem obviously holds since only one successful performance is required for all p_1. For $n > 1$ assume the theorem is false. Then there is a system with $k > 1$, $p_n < \bar{p}_n$ such that $V_n^k(p, p_n) < V_n^j(p, p_n)$ and $V_n^j(p, \bar{p}_n) < V_n^k(p, \bar{p}_n)$. Let $f_n(j,k,p)$ be as defined in Lemma 5. Then $\bar{p}_n < f_n(j,k,p)$ and $p_n > f_n(j,k,p)$ for a contradiction.

Thus, it has now been shown that for fixed $p_1, p_2, \ldots, p_{n-1}$, the value of k that minimizes $V_n^k(p)$ is a non-decreasing step function of p_n.

Of course, for $p_n = 1$, the minimizing value of k cannot exceed $V_{n-1}(p)/c_n$, since any value exceeding this would be inferior to $k = 0$.

Also, $f_n(j,k,p) \geq c_n / \left[V_{n-1}(p, 1-(1-q_n)^j (1-p_{n-1})) - V_{n-1}(p,1) \right]$.

Thus, if $V_n(p, 1-(1-q_n)^1 (1-p_{n-1})) \leq c_n + V_{n-1}(p,1)$, $f_n(j,k,p) \geq 1$ and j successful performances is preferred to k successful performances for all $k > j$. Thus, the sequence of optimal values of k in (4) is a subsequence of the set $0, 1, 2, \ldots,$ min $\{[V_{n-1}(p)/c_n], j\}$ where j is the smallest integer satisfying $V_{n-1}(p, 1-(1-q_n)^j(1-p_{n-1})) \leq c_n + V_{n-1}(p,1)$.

Theorem 7: Suppose $j < k < \ell$ and $f_n(j,k,p) > f_n(k,\ell,p)$ and $p_1, p_2, \ldots, p_{n-1}$ are fixed. Then $V_n^k(p) \neq V_n(p)$ for any p_n.

Proof: For $p_n < f_n(j,k,p)$, $V_n^j(p) < V_n^k(p)$ while for $p_n \geq f_n(j,k,p)$, $p_n > f_n(k,\ell,p)$ and $V_n^\ell(p) < V_n^k(p)$.

Theorem 8: Suppose $n(1), n(2), \ldots, n(m)$ is an increasing sequence of integers such that $f_n(n(i), n(i+1), p)$ is increasing in i and that $V_n^j(p) \neq V_n(p)$ for any p_n if j is not in this sequence. Then $V_n^{n(i)}(p) = V_n(p)$ for $f_n(n(i-1), n(i), p) < p_n \leq f_n(n(i), n(i+1), p)$.

Proof: For $p_n > f_n(n(i-1), n(i), p)$, $V_n^{n(i)}(p, p_n) < V_n^{n(i-1)}(p, p_n) < \cdots < V_n^{n(1)}(p, p_n)$ for $p_n \leq f_n(n(i), n(i+1), p)$, $V_n^{n(i)}(p, p_n) \leq V_n^{n(i+1)}(p, p_n) < \cdots < V_n^{n(m)}(p, p_n)$. Q. E. D.

This means one may start with the sequence $0,1,\ldots,$ min $\{[V_{n-1}(p)/c_n], j\}$, where j is the smallest integer such that $V_{n-1}(p, 1-(1-q_n)^j(1-p_{n-1})) \leq c_n + V_{n-1}(p)$, eliminate those members that cannot be optimal for any p_n by Theorem 7, continue to eliminate from the remaining sequence until the sequence left satisfies the conditions of Theorem 8. This procedure must be finite since only a finite number of eliminations may occur.

General Algorithm

Formally, the algorithm for finding the value of k that minimizes $V_n^k(p)$ as a function of p_n is as follows:

Algorithm 1

1. Set $n = 1$ and $V_n(p) = c_1/p_1$.

2. If $n = N$, terminate. Otherwise increase n by one and define $n(0), n(1), \ldots, n(m)$ where $n(i) = i$ and $m = \min \{[V_{n-1}(p)/c_n], j\}$ where j is the smallest integer satisfying $V_{n-1}(p, 1 - (i-q_n)^j(i-p_{n-1})) \leq c_n + V_{n-1}(p, 1)$.

3. Compute $f_n(n(i), n(i+1), p)$ for $i = 1, \ldots, k-1$ according to the formula in (8). For $n > 2$, $V_{n-1}(p)$ may be calculated by Algorithm 2.

4. If no i satisfies $f_n(n(i), n(i+1), p) > f_n(n(i+1), n(i+2), p)$ delete from the sequence any i such that $f_n(n(i), n(i+1), p) \geq 1$ and return to 2 as the value of k which minimizes $V_n(p)$

is that which satisfies $f_n(n(i-1), n(i), p) < p_n \leq f_n(n(i), n(i+1), p)$. Otherwise delete any i from the sequence satisfying $f_n(n(i), n(i+1), p) > f_n(n(i+1), n(i+2), p)$, relabel the members of the remaining sequence $n(0), n(1), \ldots, n(m)$ in increasing order with $m+1$ equaling the number of elements in the remaining sequence and return to step 3.

Given the sequences generated by algorithm 1, the optimal number of successful performances to require at each level and $V_n(p)$ for $n > 1$ may be found as follows.

Algorithm 2

1. Set $m = n$ and define $\bar{p}_n = p_n$, $k(n)$ as the $n(j)$ that satisfies $f_n(n(j-1), n(j), p) < p_n \leq f_n(n(j), n(j+1), p)$.
2. Decrease m by one.
3. Define $\bar{p}_m = 1 - (1-p_m)(1-q_{m+1})^{k(m+1)}$. Then define $k(m)$ as the $n(j)$ that satisfies $f_m(n(j-1), n(j), p) < \bar{p}_m \leq f(n(j), n(j+1), p)$.
4. If $m = 2$, define $\bar{p}_1 = 1 - (1-p_1)(1-q_2)^{k(2)}$ and go to 5. Otherwise go to 2.
5. Terminate. $V_n(p) = \sum_{i=1}^{n} k(i)c_i/\bar{p}_i$ where $k(1) = 1$.

III. ESTIMATION OF PARAMETERS

Maximum Likelihood

The past performances of students may be used to obtain a maximum likelihood estimate of the p and q input vectors.

Let a_{ki} and b_{ki} respectively be the number of incorrect and

correct responses of students at level i who had given k correct responses at level $i+1$, and define $L(p,q)$ as the likelihood function of the vector (p,q). It follows from (3) that

$$L(p,q) = \prod_{i=1}^{n} \prod_{k=0}^{m} (1-p_i)^{a_{ki}} (1-q_{i+1})^{k\, a_{ki}} \left(1-(1-p_i)(1-q_{i+1})^{k}\right)^{b_{ki}} . \qquad (9)$$

Taking the partial derivatives of $L(p,q)$ with respect to all p_i and q_i one obtains (10) and (11) setting them to zero yields (12) and (13).

$$\frac{\partial L(p,q)}{\partial p_i} = L(p,q) \sum_{i=1}^{m} \left[\frac{-a_{ki}}{1-p_1} + \frac{b_{ki}\,(1-q_{i+1})^{k}}{1-(1-p_i)(1-q_{i+1})^{k}} \right] \qquad (10)$$

$$\frac{\partial L(p,q)}{\partial q_{i+1}} = L(p,q) \sum_{k=0}^{m} \left[\frac{-k a_{ki}}{1-q_{i+1}} + \frac{k b_{ki}\,(1-p_i)(1-q_{i+1})^{k-1}}{1-(1-p_i)\,(1-q_{i+1})^{k}} \right] \qquad (11)$$

$$\sum_{k=0}^{m} a_{ki} = \sum_{k=0}^{m} \frac{b_{ki}(1-p_i)(1-q_{i+1})^{k}}{1-(1-p_i)(1-q_{i+1})^{k}} \qquad (12)$$

$$\sum_{k=0}^{m} k a_{ki} = \sum_{k=0}^{m} \frac{k b_{ki}(1-p_i)\,(1-q_{i+1})^{k}}{1-(1-p_i)(1-q_{i+1})^{k}} . \qquad (13)$$

Note that in (12) and (13) the only unknown that parameter p_i depends on is q_{i+1} and vice versa. Thus the p and q vectors may be estimated by solving sets of two simultaneous equations in two unknowns. However, there is no analytical way of solving these equations in general for p_i and q_{i+1}. Nevertheless, suppose one takes into account only the values of k with the highest number of observations. Denote these values by r and s.

Then from (12) and (13) one obtains

$$a_{ri} + a_{si} = \frac{b_{ri}(1-p_i)(1-q_{i+1})^r}{1-(1-p_i)(1-q_{i+1})^r} + \frac{b_{si}(1-p_i)(1-q_{i+1})^s}{1-(1-p_i)(1-q_{i+1})^s} \tag{14}$$

$$r\,a_{ri} + s\,a_{si} = \frac{rb_{ri}(1-p_i)(1-q_{i+1})^r}{1-(1-p_i)(1-q_{i+1})^r} + \frac{s\,b_{si}(1-p_i)(1-q_{i+1})^s}{1-(1-p_i)(1-q_{i+1})^s} . \tag{15}$$

Subtracting r times equation (14) from equation (15) yields

$$(s-r)\,a_{si} = \frac{(s-r)b_{si}(1-p_i)(1-q_{i+1})^s}{1-(1-p_i)(1-q_{i+1})^s} \tag{16}$$

which yields

$$a_{si} = \frac{b_{si}(1-p_i)(1-q_{i+1})^s}{1-(1-p_i)(1-q_{i+1})^s} \tag{17}$$

$$a_{ri} = \frac{b_{ri}(1-p_i)(1-q_{i+1})^r}{1-(1-p_i)(1-q_{i+1})^r} \quad . \tag{18}$$

From (17) and (18) one obtains

$$(1-p_i)(1-q_{i+1})^r = \frac{a_{ri}}{a_{ri} + b_{ri}} = \overline{X}_r \tag{19}$$

$$(1-p_i)(1-q_{i+1})^s = \frac{a_{si}}{a_{si} + b_{si}} = \overline{X}_s \tag{20}$$

where $\overline{X}_r$ and $\overline{X}_s$ are the proportion of unsuccessful performances at level i given r and s successful performances at level i + 1 respectively. Solving these two equations for estimates of p_i and q_{i+1} one obtains

$$p_i = 1 - \overline{X}_r^{\frac{s}{s-r}} \overline{X}_s^{\frac{-r}{s-r}} \tag{21}$$

$$q_i = 1-(\overline{X}_s/\overline{X}_r)^{\frac{1}{s-r}} \quad . \tag{22}$$

Confidence Intervals

Consider the two Bernoulli random variables, X_r and X_s, with parameters $(1-p_i)(1-q_{i+1})^r$ and $(1-p_i)(1-q_{i+1})^s$ respectively. From (19) and (20) it follows that $\overline{X}_r$ and $\overline{X}_s$ are the sample means of such random variables. Thus, it follows that for large sample sizes, a 100 $(1-\beta)$ percent confidence interval for $(1-p_i)(1-q_{i+1})^r$ is given by:

$$L_{1r} = \bar{X}_r - \frac{Z}{\sqrt{a_{ri} + b_{ri}}} \sqrt{\bar{X}_r(1-\bar{X}_r)} \tag{23}$$

$$L_{2r} = \bar{X}_r + \frac{Z}{\sqrt{a_{ri} + b_{ri}}} \sqrt{\bar{X}_r(1-\bar{X}_r)} \tag{24}$$

and for $(1-p_i)\,(1-q_{i+1})^s$ by

$$L_{1s} = \bar{X}_s - \frac{Z}{\sqrt{a_{si} + b_{si}}} \sqrt{\bar{X}_s(1-\bar{X}_s)} \tag{25}$$

$$L_{2s} = \bar{X}_s + \frac{Z}{\sqrt{a_{si} + b_{si}}} \sqrt{\bar{X}_s(1-\bar{X}_s)} \tag{26}$$

where $Z = Z_{1-\beta/2}$; that is $P[Z \leq Z_{1-\beta/2}] = 1 - \beta/2$ where Z is standard normal.*

Since the probabilities that (L_{1r}, L_{2r}) and (L_{1s}, L_{2s}) bracket $(1-p_i)\,(1-q_{i+1})^r$ and $(1-p_i)\,(1-q_{i+1})^s$ are each $1-\beta$ and independent, the probability that both of these bounds hold is $1-2\beta+\beta^2 \geq 1-2\beta$.

*Larson (1966).

From this it follows that L_{1s}/L_{2r} and L_{2s}/L_{1r} form a 100 $(1-2\beta)$ percent confidence interval for $(1-q_i)^{s-r}$. Similarly it follows that L_{1r}^s/L_{2s}^r and L_{2r}^s/L_{1s}^r form 100 $(1-\beta)$ percent confidence interval for $(1-p_i)^{s-r}$.

These sets of bounds taken to the $1/(s-r)$ power give $100(1-2\beta)^{\frac{1}{s-r}}$ percent confidence intervals for $(1-q_{i+1})$ and $(1-p_i)$ respectively. Consequently a $100(1-\alpha)$ percent confidence interval for p_i is given by

$$P_1 = 1 - (L_{2r}^s/L_{1s}^r)^{\frac{1}{s-r}} \tag{27}$$

$$P_2 = 1 - (L_{1r}^s/L_{2s}^r)^{\frac{1}{s-r}} \tag{28}$$

and for q_{i+1} by

$$Q_1 = 1 - (L_{2s}/L_{1r})^{\frac{1}{s-r}} \tag{29}$$

$$Q_2 = 1 - (L_{1s}/L_{2r})^{\frac{1}{s-r}} \tag{30}$$

where L_{1s}, L_{2s}, L_{1r}, L_{2r} are as defined in (15 - 18) and $\beta = (1-\alpha/2)^{s-r}$.

Note that the above simplifies if $s-r = 1$ and that this is likely to happen as the two most likely choices for the number of successful performances to require at a given level are likely to be consecutive integers.

A Linear Estimation Model

In this model the student is given a questionnaire to determine his level of competence. Each question is scored one or zero, depending on whether the student answers the question right or wrong. The p_i's and q_i's for a student are each assumed to be linear combinations of the scores he receives on a question. Thus,

$$p_i = \sum_j u_{ij} w_j \tag{31}$$

$$q_i = \sum_j v_{ij} w_j \tag{32}$$

where w_j is the score on question j for $j > 1$ and $w_0 \equiv 1$. This yields, for the vector pair (u, v), the likelihood function

$$L(u,v) = \prod_{i=1}^{n} \left[\prod_{\ell \in \bar{X}_i} (1 - \sum_j u_{ij} w_{j\ell}) (1 - \sum_j v_{i+1j} w_{j\ell})^k \prod_{\ell \in X_i} (1-(1- \sum_i u_{ij} w_{j\ell}) \right. \tag{33}$$

$$\left. (1 - \sum_j v_{i+1j} w_{j\ell})^k) \right]$$

where $w_{j\ell}$ is the score of student or trial ℓ on question j and

$\bar{X}_i = \{ \ell :$ student responds incorrectly at level $i \}$

$X_i = \{ \ell :$ student responds correctly at level $i \}$

Note that in (33) k is actually a function of ℓ.

Taking partial derivatives of $L(u, v)$ one obtains:

$$\frac{\partial L(u,v)}{\partial u_{ij}} = L(u,v)\left[\sum_{\ell \epsilon \bar{X}_i} \frac{-w_{j\ell}}{1-\sum_j u_{ij}w_{j\ell}} + \sum_{\ell \epsilon X_i} \frac{w_{j\ell}(1-\sum_j v_{i+1j}w_{j\ell})^k}{1-(1-\sum_j u_{ij}w_{j\ell})(1-\sum_j v_{i+1j}w_{j\ell})^k}\right] \quad (34)$$

$$\frac{\partial L(u,v)}{\partial u_{ij}} = L(u,v)\left[\sum_{\ell \epsilon \bar{X}_i} \frac{-kw_{j\ell}}{1-\sum_j v_{i+1j}w_{j\ell}} + \sum_{\ell \epsilon X_i} \frac{k(1-\sum_j u_{ij}w_{j\ell})(1-\sum_j v_{i+1j}w_{j\ell})^{k-1}}{1-(1-\sum_j u_{ij}w_{j\ell})(1-\sum_j v_{i+1j}w_{j\ell})^k}\right] \quad (35)$$

Setting both derivatives equal to zero, one obtains:

$$\sum_{\ell \epsilon \bar{X}_i} \frac{w_{j\ell}}{1-\sum_j u_{ij}w_{j\ell}} = \sum_{\ell \epsilon X_i} \frac{w_{j\ell}(1-\sum_j v_{i+1j}w_{j\ell})^k}{1-(1-\sum_j u_{ij}w_{j\ell})(1-\sum_j v_{i+1j}w_{j\ell})^k} \quad (36)$$

$$\sum_{\ell \epsilon \bar{X}_i} \frac{kw_{j\ell}}{1-\sum_j v_{i+1j}w_{j\ell}} = \sum_{\ell \epsilon X_i} \frac{kw_{j\ell}(1-\sum_j u_{ij}w_{j\ell})(1-\sum_j v_{i+1j}w_{j\ell})^{k-1}}{1-(1-\sum_j u_{ij}w_{j\ell})(1-\sum_j v_{i+1}w_{j\ell})^k} \quad (37)$$

Since different values of ℓ lead to different denominators in the terms of (36) and (37), some simplification of these expressions in needed. Let:

w_{ijk} = number of one scores for question j resulting in successful performances at level i following k successful performances at level i+1.

$\bar{w}_{ijk}$ = number of zero scores for question j resulting in successful performances at level i following k successful performances at level i+1.

w'_{ijk} = number of one scores for question j resulting in unsuccessful performances at level i following k successful performances at level i+1.

$\bar{w}'_{ijk}$ = number of zero scores for question j resulting in unsuccessful performances at level i following k successful performances at level i+1.

Suppose further that only question j is to be considered on the questionnaire. Then (36) and (37), when applied to u_{ij}, v_{i+1j}, u_{i0}, v_{i+10} become:

$$\sum_{k=0}^{m} \frac{w'_{ijk}}{1-u_{ij}-u_{i0}} = \sum_{k=0}^{m} \frac{w_{ijk}(1-v_{i+1j}-v_{i+10})^{k}}{1-(1-u_{ij}-u_{i0})\,(1-v_{i+1j}-v_{i+10})^{k}} \tag{38}$$

$$\sum_{k=0}^{m} \frac{k\,w'_{ijk}}{1-v_{i+1j}-v_{i+10}} = \sum_{k=0}^{m} \frac{k\,w_{ijk}(1-u_{ij}-u_{i0})(1-v_{i+1j}-v_{i+10})^{k-1}}{1-(1-u_{ij}-u_{i0})\,(1-v_{i+1j}-v_{i+10})^{k}} \tag{39}$$

$$\sum_{k=0}^{m} \frac{w'_{ijk}}{1-u_{ij}-u_{i0}} + \sum_{k=0}^{m} \frac{\bar{w}'_{ijk}}{1-u_{i0}} = \sum_{k=0}^{m} \frac{w_{ijk}(1-v_{i+1}-v_{i+10})^k}{1-(1-u_{ij}-u_{i0})(1-v_{i+1j}-v_{i+10})^k}$$

$$+ \sum_{k=0}^{m} \frac{\bar{w}_{ijk}(1-v_{i+10})^k}{1-(1-u_{i0})(1-v_{i+10})^k} \qquad (40)$$

$$\sum_{k=0}^{m} \frac{k\,w'_{ijk}}{1-v_{ij}-v_{i0}} + \sum_{k=0}^{m} \frac{k\,\bar{w}_{ijk}}{1-v_{i0}} = \sum_{k=0}^{m} \frac{k\,w_{ijk}(1-u_{ij}-u_{i0})(1-v_{i+1}-v_{i+10})^{k-1}}{1-(1-u_{ij}-u_{i0})(1-v_{i+1}-v_{i+10})^k}$$

$$+ \sum_{k=0}^{m} \frac{k\,\bar{w}_{ijk}(1-u_{i0})(1-v_{i+10})^{k-1}}{1-(1-u_{i0})(1-v_{i+10})^k} \qquad (41)$$

However, when (37) and (38) are multiplied by $1-u_{ij}-u_{i0}$ and $1-v_{i+1j}-v_{i+10}$ they become (12) and (13) with w'_{ijk}, w_{ijk}, $u_{ij}+u_{i0}$, $v_{i+1j}+v_{i+10}$ replacing a_k, b_{ki}, p_i, and q_{i+1} respectively.

Thus if one takes into account observations corresponding only to the two highest values of k, r and s, one obtains from (19-22).

$$u_{ij}+u_{i0} = 1 - \left(\frac{w'_{ijr}}{w_{ijr}+w'_{ijr}}\right)^{\frac{s}{s-r}} \left(\frac{w'_{ijs}}{w_{ijs}+w'_{ijs}}\right)^{\frac{-r}{s-r}} \qquad (42)$$

$$v_{i+1j} + v_{i+10} = 1 - \left[\frac{w'_{ijs}(w_{ijr} + w'_{ijr})}{w'_{ijr}(w_{ijs} + w'_{ijs})} \right]^{\frac{1}{s-r}} \qquad (43)$$

Subtracting (38) and (39) from (40) and (41) and then multiplying through by $1-u_{i0}$ and $1-v_{i+10}$ yield (12) and (13) with $\bar{w}'_{ijk}$, $\bar{w}_{ijk}$, u_{i0}, and v_{i+10} replacing a_{ki}, b_{ki}, p_i, and q_{i+1} respectively. Thus, after considering only r and s as values of k, one obtains from (19-22) and (42) and (43).

$$u_{i0} = 1 - \left(\frac{\bar{w}'_{ijr}}{\bar{w}_{ijr} + \bar{w}'_{ijr}} \right)^{\frac{s}{s-r}} \left(\frac{\bar{w}'_{ijs}}{\bar{w}_{ijs} + \bar{w}'_{ijs}} \right)^{\frac{-r}{s-r}} \qquad (44)$$

$$v_{i+10} = 1 - \left[\frac{\bar{w}'_{ijs}(\bar{w}'_{ijr} + \bar{w}_{ijr})}{\bar{w}'_{ijr}(\bar{w}'_{ijs} + \bar{w}_{ijs})} \right]^{\frac{1}{s-r}} \qquad (45)$$

$$u_{ij} = \left(\frac{\bar{w}'_{ijr}}{\bar{w}_{ijr} + \bar{w}'_{ijr}} \right)^{\frac{s}{s-r}} \left(\frac{\bar{w}'_{ijs}}{\bar{w}_{ijs} + \bar{w}'_{ijs}} \right)^{\frac{-r}{s-r}} - \left(\frac{w'_{ijr}}{w_{ijr} + w'_{ijr}} \right)^{\frac{s}{s-r}} \left(\frac{w'_{ijs}}{w_{ijs} + w'_{ijs}} \right)^{\frac{-r}{s-r}} \qquad (46)$$

$$v_{i+1j} = \left[\frac{\bar{w}'_{ijs}(\bar{w}'_{ijr} + \bar{w}_{ijr})}{\bar{w}'_{ijr}(\bar{w}'_{ijs} + \bar{w}_{ijs})} \right]^{\frac{1}{s-r}} - \left[\frac{w'_{ijs}(w_{ijr} + w'_{ijr})}{w'_{ijr}(w_{ijs} + w'_{ijs})} \right]^{\frac{1}{s-r}} \qquad (47)$$

In a similar manner one could also obtain $100(1-\alpha)$ percent confidence intervals for u_{ij}, v_{i+1j}, u_{i0}, and v_{i+10} by noting that $100(1-\alpha)$ percent confidence intervals for $u_{ij} + u_{i0}$ and $v_{i+1j} + v_{i+10}$ are also $100(1-\alpha)$ percent confidence intervals for u_{ij} and v_{i+1j} and making the appropriate substitutions in (23-30). Note that the length of these confidence intervals tends to zero as the sample size for both k=r and k=s tend to infinity and consequently the estimates in (44-47) are consistent.

In (44-47) the weights are based on the assumption that all weights except for question j are zero. For the more general case where this is not assumed, it is suggested that the weights used be the average of the results in (44-47) taken over all j. That is:

$$u_{i0} = 1 - \frac{1}{\bar{n}} \sum_j \left(\frac{\bar{w}'_{ijr}}{\bar{w}_{ijk} + \bar{w}'_{ijr}} \right)^{\frac{s}{s-r}} \left(\frac{\bar{w}'_{ijs}}{\bar{w}_{ijs} + \bar{w}'_{ijs}} \right)^{\frac{-r}{s-r}} \qquad (48)$$

$$v_{i+10} = 1 - \frac{1}{\bar{n}} \sum_j \left[\frac{\bar{w}'_{ijs}(\bar{w}'_{ijr} + \bar{w}_{ijr})}{\bar{w}'_{ijr}(\bar{w}'_{ijs} + \bar{w}_{ijs})} \right]^{\frac{1}{s-r}} \qquad (49)$$

$$u_{ij} = \frac{1}{\bar{n}} \left(\frac{\bar{w}'_{ijr}}{\bar{w}_{ijr} + \bar{w}'_{ijr}} \right)^{\frac{s}{s-r}} \left(\frac{\bar{w}'_{ijs}}{\bar{w}_{ijs} + \bar{w}_{ijs}} \right)^{\frac{-r}{s-r}} \tag{50}$$

$$- \frac{1}{\bar{n}} \left(\frac{w'_{ijr}}{w_{ijr} + w'_{ijr}} \right)^{\frac{s}{s-r}} \left(\frac{w'_{ijs}}{w_{ijs} + w'_{ijs}} \right)^{\frac{-r}{s-r}} , \; j \neq 0$$

$$v_{i+1j} = \frac{1}{\bar{n}} \left[\frac{\bar{w}'_{ijs}(\bar{w}'_{ijr} + \bar{w}_{ijr})}{\bar{w}'_{ijr}(\bar{w}'_{ijs} + \bar{w}_{ijs})} \right]^{\frac{1}{s-r}} - \frac{1}{\bar{n}} \left[\frac{w'_{ijs}(w_{ijr} + w'_{ijr})}{w'_{ijr}(w_{ijs} + w'_{ijs})} \right]^{\frac{1}{s-r}} , j \neq 0 \tag{51}$$

where $\bar{n}$ is the number of questions on the questionnaire.

IV. EXAMPLE

Consider the three level problem with the following data.

$(c_1, c_2, c_3) = (6.7, 4.3, 1.9)$

$(p_1, p_2, p_3) = (.07, .23, .46)$

$(q_2, q_3) = (.6, .8)$

It then follows that $V_1(p_1) = 6.7/p_1$.

For level 2, substitute the appropriate values for c_1, c_2, p_1, q_2 in (7) to obtain $f(k, p_1) = .995(.4)^k - 1.498 + 1.15/.4^{k-1}$

This yields $f(1, p_1) = .050$, $f(2, p_1) = 1.536$. Since the latter exceeds one, no more than one successful performance at level 2 will ever be required. This yields the following table for $V_2(p)$.

i	n(i)	f
0	0	.000
1	1	.050

Table 1: Output for level 2

This table may be read as follows. For $0 \leq p_2 \leq .050$, require no successful performances at level 2. For $p_2 > .050$, require one successful performance at level 2.

For level 3, note from (8) that

$$f_3(k-1, 1, p) = 1.9/ [V_2(p, 1 - .77(.2)^{k-1}) - V_2(p, 1-.77(.2)^k)] .$$

In order to find $f_3(0, 1, p)$, $V_2(p, 1-.77(.2)^0) = V_2(p,.23)$ and $V_2(p, 1-.77(.2)) = V_2(p, .846)$ must be calculated by algorithm 2. For the calculation of $V(p,.23)$, $k(2) = 1$ by table 1. Thus $\bar{p}_1 = 1-(.4)(.93) = .628$ and $V(p,.23) = 4.3/.23 + 6.7/.628 = 29.37$. Similarly, $V(p, .846) = 15.75$. Thus $f_3(0, 1, p) = 1.9/(29.37 - 15.75) = .140$. Similarly, $V_2(p, 1-.77(.2)^2) = V_2(p,.969) = 15.11$ and $f(1, 2, p) = 2.97$. For $k > 3$, $V_2(p, 1 - .77(.2)^{k-1}) \leq V_2(p,.969) = 15.11$ and $V_2(p, 1-.77(.2)^k) \geq V_2(p, 1) = 14.97$. Thus

$f(k-1,k,p) \geq 4.3/(15.11 - 14.97) = 30.71 > 1$. Thus no more than 1 successful performance at level 3 can be required, yielding the following table.

i	n(i)	f
0	0	.000
1	1	.140

Table 2: Output for level 3.

Table 1 and 2 contain the information necessary to use algorithm 2 to calculate $V(.07,.23,.46)$. From table 2, $k(3) = 1$ since $.140 \leq .46 < 1$. Thus $\bar{p}_2 = 1 - .77(.2) = .846$. From table 1, $k(2) = 1$ since $.140 < .628 < 1$ and $\bar{p}_1 = 1 - .93(.4) = .628$. Thus one successful performance is required at each level and the expected time to complete the course is $V(.07, .23,.46) = 6.7/.628 + 4.3/.846 + 1.9/.46 = 19.88$.

List of Symbols

$p = (p_1, \ldots, p_N)$	state vector where p_i is the probability student can perform at level i.
q_i	probability student can perform at level i-1 given that he performs successfully at level i and could not previously perform successfully at level i - 1.
$V(\pi, p)$	expected cost under policy π when the system is in state p.
$V_n(p)$	minimum cost for state p if we restrict instruction to levels 1, ..., n.
$V_n^k(p)$	same as $V_n(p)$ except that exactly k successful performances at level n are required.
$f(k, p_1)$	the value of p_n such that $V_2^k(p) = V_2^{k-1}(p)$ for fixed p_1.
$f_n(j, k, p)$	the value of p_n such that $V_n^j(p) = V_n^k(p)$, $j < k$.
a_{ki}	number of incorrect responses at level i following k correct responses at level i + 1.
b_{ki}	number of correct responses at level i following k correct responses at level i + 1.
$L(p, q)$	the likelihood function of the vector (p, q).
w_j	score on question j.
u_{ij}	weight of question j upon p_i.
v_{ij}	weight of question j upon q_i.
w_{ijk}	as defined in text.
$\bar{w}_{ijk}$	as defined in text.
w_{ijk}	as defined in text.
$\bar{w}_{ijk}$	as defined in text.

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Personnel & Training Research Programs
Office of Naval Research
Arlington, VA 22217

1 Director
ONR Branch Office
495 Summer Street
Boston, MA 02210
ATTN: C. M. Harsh

1 Director
ONR Branch Office
1030 East Green Street
Pasadena, CA 91101
ATTN: E. E. Gloye

1 Director
ONR Branch Office
536 South Clark Street
Chicago, IL 60605
ATTN: M. A. Bertin

6 Director
Naval Research Laboratory
Code 2627
Washington, DC 20390

12 Defense Documentation Center
Cameron Station, Building 5
5010 Duke Street
Alexandria, VA 22314

1 Chairman
Behavioral Science Department
Naval Command & Management Division
U.S. Naval Academy
Luce Hall
Annapolis, MD 21402

1 Chief of Naval Technical Training
Naval Air Station Memphis (75)
Millington, TN 38054
ATTN: Dr. N. Kerr

1 Chief of Naval Training
Naval Air Station
Pensacola, FL 32508
ATTN: CAPT Bruce Stone, USN

1 LCDR Charles J. Theisen, Jr., MSC, USN 4024
Naval Air Development Center
Warminster, PA 18974

1 Commander
Naval Air Reserve
Naval Air Station
Glenview, IL 60026

1 Commander
Naval Air Systems Command
Department of the Navy
AIR-413C
Washington, DC 20360

1 Mr. Lee Miller (AIR 413E)
Naval Air Systems Command
5600 Columbia Pike
Falls Church, VA 22042

1 Dr. Harold Booher
NAVAIR 415C
Naval Air Systems Command
5600 Columbia Pike
Falls Church, VA 22042

1 CAPT John F. Riley, USN
Commanding Officer
U.S. Naval Amphibious School
Coronado, CA 92155

1 Special Assistant for Manpower
OASN (M&RA)
The Pentagon, Room 4E794
Washington, DC 20350

1 Dr. Richard J. Neihaus
Office of Civilian Manpower Management
Code 06A
Department of the Navy
Washington, DC 20390

1 CDR Richard L. Martin, USN
COMFAIRMIRAMAR F-14
NAS Miramar, CA 02145

1 Research Director, Code 06
Research and Evaluation Dept.
U.S. Naval Examining Center
Great Lakes, IL 60088
ATTN: C. S. Winiewicz

1 Chief
Bureau of Medicine and Surgery
Code 413
Washington, DC 20372

1 Program Coordinator
Bureau of Medicine & Surgery
(Code 71G)
Department of the Navy
Washington, DC 20372

1 Commanding Officer
Naval Medical Neuropsychiatric
Research Unit
San Diego, CA 92152

1 Dr. John J. Collins
Chief of Naval Operations (OP-987F)
Department of the Navy
Washington, DC 20350

1 Technical Library (Pers-11B)
Bureau of Naval Personnel
Department of the Navy
Washington, DC 20360

10 Dr. James J. Regan
Navy Personnel Research &
Development Center
San Diego, CA 92152

1 Commanding Officer
Navy Personnel Research
and Development Center
San Diego, CA 92152

1 Superintendent
Naval Postgraduate School
Monterey, CA 92940
ATTN: Library (Code 2124)

1 Mr. George N. Graine
Naval Ship Systems Command
(SHIPS 03H)
Department of the Navy
Washington, DC 20360

1 Technical Library
Naval Ship Systems Command
National Center, Building 3
Room 3S08
Washington, DC 20360

1 Commanding Officer
Service School Command
U.S. Naval Training Center
San Diego, CA 92133
ATTN: Code 303

1 Chief of Naval Training Support
Code N-21
Building 45
Naval Air Station
Pensacola, FL 32508

1 Dr. William L. Maloy
Principal Civilian Advisor
for Education & Training
Naval Training Command
Code 01A
Pensacola, FL 32508

1 Dr. Hanss H. Wolff
Technical Director (Code N-2)
Naval Training Equipment
Center
Orlando, FL 32813

1 Mr. Arnold Rubinstein
Naval Material Command
(NMAT-03424)
Room 820, Crystal Plaza #6
Washington, DC 20360

Dr. H. Wallace Sinaiko
c/o Office of Naval Research
(Code 450)
Psychological Sciences Division
Arlington, VA 22217

1 Dr. Martin F. Wiskoff
Navy Personnel Research and
Development Center
San Diego, CA 92152

1 Dr. John Ford, Jr.
Navy Personnel Research and
Development Center
San Diego, CA 92152

1 Technical Library
Navy Personnel Research and
Development Center
San Diego, CA 92152

Army

1 Commandant
U.S. Army Institute of
Administration
ATTN: EA
Fort Benjamin Harrison, IN 46216

1 Armed Forces Staff College
Norfolk, VA 23511
ATTN: Library

1 Director of Research
U.S. Army Armor Human Research Unit
ATTN: Library
Building 2422 Morade Street
Fort Knox, KY 40121

1 U.S. Army Research Institute for
the Behavioral & Social Sciences
1300 Wilson Boulevard
Arlington, VA 22209

1 Commanding Officer
ATTN: LTC Montgomery
USACDC- PASA
Ft. Benjamin Harrison, IN 46249

1 Dr. John L. Kobrick
Military Stress Laboratory
U.S. Army Research Institute
of Environmental Medicine
Natick, MA 01760

1 U.S. Army Research Institute
Commonwealth Building, Room 239
1300 Wilson Boulevard
Arlington, VA 22209
ATTN: Dr. R. Dusek

1 Mr. Edmund F. Fuchs
U.S. Army Research Institute
1300 Wilson Boulevard
Arlington, VA 22209

1 Chief, Unit Training & Educational
Technology Systems
U.S. Army Research Institute for
the Behavioral and Social
Sciences
1300 Wilson Boulevard
Arlington, VA 22209

1 Commander
U.S. Theater Army Support
Command, Europe
ATTN: Asst. DCSPER (Education)
APO New York 09058

1 Dr. Stanley L. Cohen
Work Unit Area Leader
Organizational Development Work
Unit
Army Research Institute for
Behavioral and Social Science
1300 Wilson Boulevard
Arlington, VA 22209

1 Dr. Leon H. Nawrocki
U.S. Army Research Institute
Rosslyn Commonwealth Building
1300 Wilson Boulevard
Arlington, VA 22209

Air Force

1 Dr. Martin Rockway
Technical Training Division
Lowry Air Force Base
Denver, CO 80230

1 Maj. P. J. DeLeo
Instructional Technology Branch
AF Human Resources Laboratory
Lowry AFB, Co 80230

1 Headquarters, U.S. Air Force
Chief, Personnel Research &
Analysis Division (AF/DPSY)
Washington, DC 20330

1 Research and Analysis Division
AF/DPXYR Room 4C200
Washington, DC 20330

1 AFHRL/AS (Dr. G.A. Eckstrand)
Wright-Patterson AFB
Ohio 45433

1 AFHRL/MD
701 Prince Street
Room 200
Alexandria, VA 22314

1 AFOSR (NL)
1400 Wilson Boulevard
Arlington, VA 22209

1 Commandant
USAF School of Aerospace Medicine
Aeromedical Library (SUL-4)
Brooks AFB, TX 78235

1 CAPT Jack Thorpe, USAF
Department of Psychology
Bowling Green State University
Bowling Green, OH 43403

1 Headquarters Electronic Systems Division
ATTN: Dr. Sylvia R. Mayer/MCIT
LG Hanscom Field
Bedfor, MA 01730

1 Lt. Col. Henry L. Taylor, USAF
Military Assistant for Human Resources
OAD (E&LS) ODDR&E
Pentagon, Room 3D129
Washington, DC 20301

Marine Corps

1 COL George Caridakis
Director
Office of Manpower Utilization
Headquarters, Marine Corps (A01H)
MCB
Quantico, VA 22134

1 Dr. A. L. Slafkosky
Scientific Advisor (Code Ax)
Commandant of the Marine Corps
Washington, DC 20380

1 Mr. E. A. Dover
Manpower Measurement Unit
(Code A01M-2)
Arlington Annex, Room 2413
Arlington, VA 20370

Coast Guard

1 Mr. Joseph J. Cowan, Chief
Psychological Research Branch(P-1)
U.S. Coast Guard Headquarters
400 Seventh Street, SW
Washington, DC 20590

Other DOD

1 Lt. Col. Austin W. Kibler, Director
Human Resources Research Office
Advanced Research Projects Agency
1400 Wilson Boulevard
Arlington, VA 22209

1 Mr. Helga Yeich, Director
Program Management, Defense Advanced Research Projects Agency
1400 Wilson Boulevard
Arlington, VA 22209

1 Mr. William J. Stormer
DOD Computer Institute
Washington Navy Yard
Building 175
Washington, DC 20374

Other Government

1 Office of Computer Information
Institute for Computer Sciences and Technology
National Bureau of Standards
Washington, DC 20234

1 Dr. Eric McWilliams Program Manager
Technology and Systems, TIE
National Science Foundation
Washington, DC 20550

Miscellaneous

1 Dr. Scarvia B. Anderson
Educational Testing Service
17 Executive Park Drive, N.E.
Atlanta, GA 30329

1 Dr. Richard C. Atkinson
Stanford University
Department of Psychology
Stanford, CA 94305

1 Mr. Edmund C. Berkeley
Berkeley Enterprises, Inc.
815 Washington Street
Newtonville, MA 02160

1 Dr. Ronald P. Carver
American Institutes for Research
8555 Sixteenth Street
Silver Spring, MD 20910

1 Dr. Allan M. Collins
Bolt Beranek and Newman
50 Moulton Street
Cambridge, MA 02138

1 Dr. Rene' V. Dawis
University of Minnesota
Department of Psychology
Minneapolis, MN 55455

2 ERIC
Processing and Reference Facility
4833 Rugby Avenue
Bethesda, MD 20014

1 Dr. Victor Fields
Department of Psychology
Montgomery College
Rockville, MD 20850

1 Dr. Edwin A. Fleishman
American Institutes for Research
8555 Sixteenth Street
Silver Spring, MD 20910

1 Dr. Duncan N. Hansen
Mempis State University
Bureau of Educational Research and Services
Memphis, TN 38152

1 Dr. Robert Glaser, Director
University of Pittsburgh
Learning Research & Development Center
Pittsburgh, PA 15213

1 Dr. Henry J. Hamburger
University of California
School of Social Sciences
Irvine, CA 92664

1 Human Resources Research Organization
Division #3
P. O. Box 5787
Presidio of Monterey, CA 93940

1 Dr. Lawrence B. Johnson
Lawrence Johnson and Associates, Inc.
200 S Street, N.W., Suite 502
Washington, DC 20009

1 Dr. David Klahr
Carnegie-Mellon University
Graduate School of Industrial Admin.
Pittsburgh, PA 15213

1 Dr. Robert R. Mackie
Human Factors Research, Inc.
6780 Cortona Drive
Santa Barbara Research Park
Goleta, CA 93017

1 Dr. Andrew R. Molnar
Technological Innovations in Education
National Science Foundation
Washington, DC 20550

1 Dr. Leo Munday
Vice President
American College Testing Program
P. O. Box 168
Iowa City, IA 52240

1 Dr. Donald A. Norman
University of California, San Diego
Center for Human Information Processing
La Jolla, CA 92037

1 Dr. Diane M. Ramsey-Klee
R-K Research &System Design
3947 Ridgemont Drive
Malibu, CA 90265

1 Dr. Leonard L. Rosenbaum
Chairman
Department of Psychology
Montgomery College
Rockville, MD 20850

1 Dr. George E. Rowland
Rowland and Company, Inc.
P. O. Box 61
Haddonfield, NJ 08033

1 Dr. Arthur I. Siegel
Applied Psychological Services
Science Center
404 East Lancaster Avenue
Wayne, PA 19087

1 Mr. Dennis J. Sullivan
725 Benson Way
Thousand Oaks, CA 91360

1 Dr. Benton J. Underwood
Northwestern University
Department of Psychology
Evanston, IL 60201

1 Dr. David J. Weiss
University of Minnesota
Department of Psychology
Minneapolis, MN 55455

1 Dr. Anita West
Denver Research Institute
University of Denver
Denver, CO 80210

1 Dr. Kenneth Wexler
University of California
School of Social Sciences
Irvine, CA 92664

1 Dr. John Annett
The Open University
Milton Keynes
Buckinghamshire
ENGLAND

1 Dr. Milton S. Katz
MITRE Corporation
Westgate Research Center
McLean, VA 22101

1 Dr. Charles A. Ullmann
Director
Behavioral Sciences Studies
Information Concepts, Inc.
1701 N. Ft. Myer Drive
Arlington, VA 22209

1 Dr. Dexter Fletcher
University of Illinois at Chicago Circle
Department of Psychology
Box 4348
Chicago, IL 60680

1 Dr. Alfred F. Smode
Staff Consultant
Training Analysis and Evaluation Group
Naval Training Equipment Center
Code N-OOT
Orlando, FL 32813

1 U.S.C. Information Sciences Inst.
Library
4676 Admiralty Way
Marina Del Rey, CA 90291

1 Mr. Rich Hale
U.S.C. Information Sciences Inst.
4676 Admiralty Way
Marina Del Rey, CA 90291

2 Jaclyn Caselli
Director of Acquisitions
ERIC Clearinghouse on Educational Media and Technology
507 Stanford Center for R&D in Teaching
Stanford University
Stanford, CA 94305